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Canonical Quantization of Inhomogeneous Strings

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The canonical quantization procedure for homogeneous nonrelativistic strings is extended to inhomogeneous strings having nonconstant density and tension, in fixed-, free-, and closed-boundary configurations. The general oscillator expansion form of the Hamiltonian is maintained. Possibilities of harmonic spectra and corresponding interactions are explored.

1. INTRODUCTION

Current activity in the theory of strings and superstrings as models for unified theories of particles and fields and their quantization (Moffat, 1986; Mansfield, 1987) makes it worthwhile to investigate extensions of the procedure for quantizing ordinary (nonrelativistic) strings. Cohen-Tannoudji *et al.* (1977, p. 611) mention the quantization of homogeneous strings, and Fogleman (1987) provides an extended presentation.

In this paper, I demonstrate that, with appropriately defined raising and lowering operators, the canonical quantization procedure may be extended, and is powerful enough to accommodate the normal mode development of the Hamiltonian formalism even for inhomogeneous strings with spatially-varying density and tension and with fixed-, free-, or closed-boundary conditions. The procedure broadly parallels the homogeneous case, but care must be exercised in the choice of definition of the position and momentum operators.

I also explore the possibility of "interactions" through joining of homogeneous strings of different densities in such a way that the property of a harmonic spectrum is preserved.

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2. THE CLASSICAL INHOMOGENEOUS STRING

The vibrations of the general classical inhomogeneous string of length L with mass per unit length $\rho(x)$ and tension $T(x)$ are described by the differential equation (Meirovitch, 1975, Chapter 5)

$$\frac{d}{dx}\left[T(x)\frac{du}{dx}\right] = -\omega^2\rho(x)u(x), \qquad 0 \leq x \leq L \tag{2.1}$$

where the transverse displacement string amplitude is given by

$$y(x, t) = \exp(-i\omega t)\, u(x) \tag{2.2}$$

and ω is the angular frequency of vibration.

For fixed-boundary conditions

$$u(0) = 0 = u(L) \tag{2.3a}$$

and also for free, i.e., open-boundary conditions

$$u'(0) = 0 = u'(L) \tag{2.3b}$$

as well as for periodic conditions $u(x) = u(x+L)$, i.e., closed-boundary conditions [with $T(0) = T(L)$]

$$u(0) = u(L), \qquad u'(0) = u'(L) \tag{2.3c}$$

the eigenfunctions $u_n(x)$ of (2.1) with angular eigenfrequencies ω_n satisfy the pair of weighted orthogonality conditions

$$\int_0^L \rho(x)u_n(x)u_m(x)\, dx = N_n\delta_{nm} \tag{2.4a}$$

$$\int_0^L T(x)u'_n(x)u'_m(x)\, dx = \omega_n^2 N_n\delta_{nm} \tag{2.4b}$$

Here, N_n are the normalization constants.

Any function $F(x)$ may then be expanded as

$$F(x) = \sum_n c_n u_n(x) \tag{2.5a}$$

with coefficients

$$c_n = \frac{1}{N_n}\int_0^L F(x)\rho(x)u_n(x)\, dx \tag{2.5b}$$

3. THE QUANTIZED INHOMOGENEOUS STRING

The position operator $Y(x, t)$ and canonically conjugate momentum operator $\Pi(x, t)$ are required to satisfy the canonical equal-time commutation relations

$$[Y(x, t), Y(x', t)] = 0 = [\Pi(x, t), \Pi(x', t)] \tag{3.1a}$$

$$[\Pi(x, t), Y(x', t)] = -i\hbar\delta(x - x') \tag{3.1b}$$

We seek an operator formalism which yields the standard commutation relations for the lowering- and raising-type operators which are introduced. The appropriate formulation for the inhomogeneous string, which effectively reduces to the standard quantization formalism in the constant-ρ, T case of Fogleman (1987), is found to be given by

$$A_n(t) = \frac{1}{N_n^{1/2}}\left(\frac{\omega_n}{2\hbar}\right)^{1/2} \int_0^L \left[\rho(x)\, Y(x, t) + \frac{i}{\omega_n}\Pi(x, t)\right] u_n(x)\, dx \tag{3.2}$$

with the corresponding expression for $A_n^\dagger$.

Then, by equations (2.5),

$$Y(x, t) = \sum_n \frac{1}{(2N_n)^{1/2}}\left(\frac{\hbar}{\omega_n}\right)^{1/2} [A_n(t) + A_n^\dagger(t)]u_n(x) \tag{3.3a}$$

$$\Pi(x, t) = \sum_n \frac{-i}{(2N_n)^{1/2}} (\hbar\omega_n)^{1/2}[A_n(t) - A_n^\dagger(t)]u_n(x) \tag{3.3b}$$

Note that equations (3.2), (3.3) are defined for any spectrum $\{\omega_n\}$, not necessarily harmonic in general. Also, equations (3.3) do not contain $\rho(x)$ explicitly, a feature which is not immediately evident from inspection of the homogeneous case.

Upon use of equation (2.4a) for fixed, open, or closed boundary conditions, this formalism for $A_n(t)$ guarantees that these operators satisfy the standard commutation relations

$$[A_n(t), A_m^\dagger(t)] = \delta_{nm} \tag{3.4}$$

for *arbitrary* functions $\rho(x)$ and $T(x)$.

The quantum Hamiltonian is now

$$H = \int_0^L \left\{\frac{1}{2\rho(x)}[\Pi(x, t)]^2 + \frac{1}{2}\, T(x)\left(\frac{\partial Y}{\partial x}\right)^2\right\} dx \tag{3.5}$$

Upon use of equations (2.4a), (2.4b), (3.3), and (3.4), this leads to

$$H = \sum_n \hbar\omega_n(A_n^\dagger A_n + \tfrac{1}{2}) \tag{3.6}$$

The quantum Hamiltonian for the inhomogeneous string is thus just the sum of harmonic oscillator Hamiltonians as for the homogeneous string, but now with frequencies ω_n which represent the spectrum of the corresponding classical inhomogeneous string (2.1) and which therefore need *not* be integer multiples of ω_1.

One may then ask: when does the Hamiltonian (3.6) equal

$$H = \sum_n \hbar(n\omega_1)(A_n^\dagger A_n + \tfrac{1}{2}) \tag{3.7}$$

as in the homogeneous case with equally-spaced oscillator frequencies, so that the subsequent formalism may then formally be discussed in the manner of Fogleman (1987)? This occurs in the interesting cases of inhomogeneous strings with density functions which nevertheless yield exactly harmonic spectra $\omega_n = n\omega_1$, some of which will now be described.

4. HARMONIC INHOMOGENEOUS CASES

For continuous density distributions, Borg (1946, Section 20) showed that a harmonic spectrum is retained for inverse fourth-power density, $\rho \sim 1/(A+Bx)^4$, for a string with fixed ends. Cases of stepped strings, with one or more jumps in otherwise constant density values, have been discovered (Gottlieb, 1986, 1987) for fixed and free configurations when an appropriate relation between density ratio and junction location is satisfied, leading to completely harmonic spectra. The cases may even be combined (Gottlieb, 1987) to give a fixed string with stepped, inverse fourth-power density having a harmonic spectrum.

The above instances are in the classical configurations of fixed of free ends. In view of the interest also in closed string field theories (Mansfield, 1987), let us investigate here the stepped *closed* string. Suppose that

$$\rho(x) = \rho_1, \qquad 0 \le x \le \alpha L \tag{4.1a}$$

$$= \rho_2, \qquad \alpha L < x \le L \tag{4.1b}$$

where $0 < \alpha < 1$ (and T is constant). We find that, with boundary conditions (2.3c), and using continuity of displacement and slope at $x = \alpha L$, if the condition

$$(\rho_1/\rho_2)^{1/2} = (1-\alpha)/\alpha \tag{4.2}$$

is satisfied, then the spectrum is purely harmonic, and the doubly-degenerate positive frequencies are

$$\omega_n^{(s)} = \omega_n^{(c)} = n\pi(T/\rho_1)^{1/2}/\alpha L; \qquad n = 1, 2, \ldots \tag{4.3}$$

(with $\omega_0 = 0$ and $u_0 = \text{const}$). The corresponding normalized orthogonal eigenfunctions [$N_n \equiv 1$ in (2.4a)] may be taken as the pairs

$$u_{n1}^{(s)} = a \sin(n\pi x/\alpha L)$$

$$u_{n2}^{(s)} = -[(1-\alpha)/\alpha]a \sin[n\pi(L-x)/(1-\alpha)L], \qquad a = (2/\rho_1 L)^{1/2} \tag{4.4a}$$

$$u_{n1}^{(c)} = b \cos(n\pi x/\alpha L)$$

$$u_{n2}^{(c)} = b \cos[n\pi(L-x)/(1-\alpha)L], \qquad b = [(1-\alpha)/\alpha]^{1/2} a \tag{4.4b}$$

There is an extra factor of 2 in the Hamiltonian (3.7) if the sum there is now taken over distinct harmonic ω.

5. DISCUSSION

In relativistic field theories, interactions between strings may be considered to take place by breaking and recombination (Witten, 1986). At the simpler level of this paper, we may consider what happens with quantized nonrelativistic strings under such juxtapositions.

If two different homogeneous open strings are joined, with continuity of amplitude and slope at the junction, an inhomogeneous open string results. This "stepped" string will again be quantizable with *harmonic* frequency spectrum if the constant densities of the original constituent strings and their lengths are in the correct relationship as specified by equations (4.1), (4.2) above. Higher multiplicities of constituent strings of different constant densities which combine to give an inhomogeneous stepped string which nevertheless possesses an exactly harmonic spectrum may also be elucidated (Gottlieb, 1987).

It also follows from the analysis presented at the end of Section 4 that two different homogeneous strings may join at both ends to form a *closed* stepped inhomogeneous string which is quantizable with (doubly-degenerate) harmonic spectrum when their densities and lengths are appropriately related, via (4.2).

Thus, suitable unequal homogeneous strings of different densities may "interact" through joining at one or both ends to produce an inhomogeneous string which retains the property of a harmonic spectrum.

One can even envisage an ensemble of finite homogeneous free strings (under equal tensions) with different densities and lengths but common product $\rho^{1/2}l$, thus sharing a common harmonic spectrum. Any two of these may interact by joining at both ends to form a closed stepped string which, by (4.2) and (4.3), has the *same* (now doubly-degenerate) harmonic excitation spectrum. Two such composite closed strings may further interact with each other by exchanging one of their homogeneous constituents to form a different pair of closed stepped strings. Throughout such operations the common, harmonic, spectrum is maintained for the quantization procedures.

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REFERENCES

Borg, G. (1946). Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe, *Acta Mathematica*, **78**, 1-96.

Cohen-Tannoudji, C., Diu, B., and Laloë, F. (1977). *Quantum Mechanics*, Volume I, Hermann, Paris.

Fogleman, G. (1987). *American Journal of Physics*, **55**, 330-336.

Gottlieb, H. P. W. (1986). *Journal of Sound and Vibration*, **108**, 63-72.

Gottlieb, H. P. W. (1987). *Journal of Sound and Vibration*, **118**, 283-290.

Mansfield, P. (1987). *Annals of Physics*, **180**, 330-352.

Meirovitch, L. (1975). *Elements of Vibration Analysis*, McGraw-Hill, New York.

Moffat, J. W. (1986). *Canadian Journal of Physics*, **64**, 561-578.

Witten, E. (1986). Mini-introduction to strings, in G. W. Gibbons, S. W. Hawking, and P. K. Townsend, eds., *Supersymmetry and its Applications*, Cambridge University Press, Cambridge, pp. 91-108.